A co-free construction for elementary doctrines

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Abstract

We provide a co-free construction which adds elementary structure to a primary doctrine. We show that the construction preserves comprehensions and all the logical operations which are in the starting doctrine, in the sense that it maps a first order many-sorted theory into a the same theory formulated with equality. As a corollary it forces an implicational doctrine to have an extentional entailment.

Introduction

This paper deals with the notion of internal equality in doctrines. Doctrines were introduced by Lawvere ([7], [6] and [8]) and we pospone in section 1 their formal definition. For the purpose of this introduction it is enough to think of doctrines as those presheaves such that, given a theory $\mathcal{T}$ over a many-sorted relational language $\mathcal{L}$, one looks at objects and morphism of the domain category as types and terms of $\mathcal{L}$ respectively, while a well formed formula in $\mathcal{T}$ of type A is an element in the fiber over A. Lawvere made extensive use of the language of adjoints and Jacobs [3] described equality between terms of a given type as a formula in the fiber over the product of that type with itself, satisfying the following rule of inference

$$\frac{\Gamma, x\colon X \mid \phi \vdash \psi[x/y]}{\Gamma, x\colon X, y\colon X \mid \phi \wedge x =_X y \vdash \psi}$$

where the double line indicates that one of the two sequents holds exactly when the other holds. A doctrine is a first order theory with equality if it possesses a formula $=_X$, for every sort X, which satisfies the previous rule.
A way to introduce higher order quantification is to consider a new type Ω in the underlying signature and thinks of terms of type Ω as propositions. From the categorical viewpoint this generates a correspondence between terms of type Ω and formulas, and therefore it makes sense to investigate how the notion of internal equality $=_\Omega$ is related to logical equivalence. A link is in the following rule, taken from [1] and [3]

$$\frac{\Gamma \mid \xi \wedge \phi \vdash \psi \qquad \Gamma \mid \xi \wedge \psi \vdash \phi}{\Gamma \mid \xi \vdash \phi =_\Omega \psi}$$

where it is implicit that if ϕ and ψ are formulas over the context Γ, then $\phi =_\Omega \psi$ is still a formula over Γ. We say that a doctrine is a higher order many-sorted

theory with extentional entailment if there is an object Ω in the base category and a formula $=_\Omega$ in the fiber over $\Omega \times \Omega$ which satisfies both the previous rules. In the present paper we provide a co-free construction that, starting from any doctrine P, produces a new doctrine $P_{\mathcal{D}}$ with equality. That is to say that for every object X in the domain category of $P_{\mathcal{D}}$ there exists a well formed formula in $P_{\mathcal{D}}(X \times X)$ which satisfies the first one of the previous rules. We show also that if the starting doctrine P is an higher order implicational theory, the resulting doctrine $P_{\mathcal{D}}$ will have an internal equality over Ω satisfying both the previous rules; in other words: $=_\Omega$ and logical equivalence comes to coincides. In section 1 we give the definitions of doctrines and some relevant examples. In section 2 we introduce the construction of Maietti and Rosolini of the category of quotients and the doctrine of descent data which is the base of the co-free construction we are going to provide in 4. In the last section we show which properties are preserved by the construction and some applications.

1 Doctrines

We recall those structures which we will be concerned with in the paper, see [10] and [11].

Definition 1.1. A **primary doctrine** is a functor $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, where **ISL** is the subcategory of **Posets** consisting of inf-semilattices and homomorphisms and $\mathbb{C}$ is a category with binary products.

For the rest of the paper we will write f^* instead of $P(f)$, to indicate the action of the functor P on a morphism f of $\mathbb{C}$. We shall refer to f^* as the reindexing functor along f. Left and right adjoints to reindexing functor f^* will be $\exists_f$ and $\forall_f$ respectively. We say that a doctrine has finite joins if every fiber has finite joins. Analogously we say that a doctrine is implicational if every fiber has relative pseudo complements which commute with reindexing. For every pair of element x and y we will denote their meet by $x \wedge y$, by $x \vee y$ their join and by $x \Rightarrow y$ their relative pseudo complements. Top and bottom elements will be $\top$ and $\bot$ respectively. Joins are said to be distributive if for every x, y and z it holds that $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$.

Definition 1.2. A primary doctrine P is said to be **elementary** if for every A in $\mathbb{C}$ there exists an object δ_A in $P(A \times A)$ such that for every X in $\mathbb{C}$

i) the assignment $\pi_1^*(\alpha) \wedge \delta_A$ determines a left adjoint to Δ_A^*

ii) the assignment $\langle \pi_1, \pi_2 \rangle^*(\alpha) \wedge \langle \pi_2, \pi_3 \rangle^*(\delta_A)$ determines a left adjoint to $(id_X \times \Delta_A)^*$

Primary doctrines are the objects of the 2-category **PD** in which

the **1-arrows** are pairs $(F, f)\colon P \longrightarrow R$

$$
\begin{tikzcd}
\mathbb{C}^{op} \arrow[dd, "F"'] \arrow[dr, "P"{name=P}] & \\
& \mathbf{ISL} \\
\mathbb{D}^{op} \arrow[ur, "R"'{name=R}] &
\arrow[from=P, to=R, "f"']
\end{tikzcd}
$$

where the functor F preserves products and f is a natural transformation from the functor $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$ to the functor $R \circ F\colon \mathcal{D}^{op} \longrightarrow \mathbf{ISL}$

the **2-arrows** are those natural transformations ν

$$
\begin{tikzcd}
\mathbb{C}^{op} \arrow[dd, bend right=40, "F"'{name=F}] \arrow[dd, bend left=40, "G"{name=G}] \arrow[dr, "P"{name=P}] & \\
& \mathbf{ISL} \\
\mathbb{D}^{op} \arrow[ur, "R"'{name=R}] &
\arrow[from=F, to=G, "\nu"]
\arrow[from=P, to=R, bend right, "f"']
\arrow[from=P, to=R, bend left, "g"]
\arrow[from=P, to=R, phantom, "\leq"]
\end{tikzcd}
$$

such that, for every object A in $\mathbb{C}$ and every α in $P(A)$, it holds that $\nu_A^*(f_A(\alpha)) \leq g_A(\alpha)$.

We call **ED** the 2-subcategory of **PD**, in which the object are elementary doctrines and the 1-arrows are those 1-arrows in **PD** such that

$$f_{A\times A}(\delta_A) = \langle F\pi_1, F\pi_2\rangle^* \delta_{FA}$$

for every 1-arrows (F, f) and for every object A in $\mathbb{C}$.

Definition 1.3. A primary doctrine is called **universal** if for every projection arrows π in $\mathbb{C}$ the functor π^* has a right adjoint $\forall_\pi$ satisfying Beck-Chevalley condition: given a pullback diagram of the kind

$$
\begin{tikzcd}
X \times Y' \arrow[r, "\pi'"] \arrow[d, "id\times f"'] & Y' \arrow[d, "f"] \\
X \times Y \arrow[r, "\pi"'] & Y
\end{tikzcd}
$$

it holds that $\forall_{\pi'} \circ (id \times f)^* = f^* \circ \forall_\pi$

A primary docrine is **existential** if the reindexing functors along a projection have a left adjoint satisfying Beck-Chevalley and Frobenius reciprocity: $\exists_\pi(\alpha \wedge \pi^*\beta) = \exists_\pi(\alpha) \wedge \beta$, for α in $P(X \times Y)$ and β in $P(Y)$.

Remark 1.4. Recall from [6, 11] that in an elementary existential doctrine P for every morphism $f\colon A \longrightarrow B$ in the base category there exists a functor $\exists_f\colon P(B) \longrightarrow P(A)$ such that $\exists_f \dashv f^*$. Indeed if π_A and π_B are the projections from $A \times B$ to A and B respectively, for α in $P(A)$

$$\exists_f(\alpha) \colon= \exists_{\pi_B}((id_B \times f)^*\delta_B \wedge \pi_A^*\alpha)$$

Such a generalized quantification satisfies Frobenius Reciprocity. For β in $P(B)$, we have that $(id_B \times f)^*\delta_B \wedge f^*\beta = (id_B \times f)^*\delta_B \wedge \pi_B^*\beta$. Therefore

$$\begin{gathered}\exists_{\pi_B}((id_B \times f)^*\delta_B \wedge \pi_A^*\alpha \wedge f^*\beta) = \\ \exists_{\pi_B}((id_B \times f)^*\delta_B \wedge \pi_A^*\alpha \wedge \pi_B^*\beta) = \\ \exists_{\pi_B}((id_B \times f)^*\delta_B \wedge \pi_A^*\alpha) \wedge \beta\end{gathered}$$

And for a pullback square such as that in 1.3 the Beck-Chevalley condition holds: $\exists_{(id_X \times f)}\ \pi^* = \pi'^*\ \exists_f$.

Definition 1.5. A primary doctrine is said to have a **weak power objects** if for every A in $\mathbb{C}$ there exists an object πA in $\mathbb{C}$ and an element $\in_A$ in $P(A \times \pi A)$ such that, for every object B in $\mathbb{C}$ and element ϕ in $P(A \times B)$ there exists a morphism $\{\phi\}\colon B \longrightarrow \pi A$ such that $\phi = (id_A \times \{\phi\})^* \in_A$.

Remark 1.6. In the case the base category $\mathbb{C}$ has a terminal object 1: the first item in the definition 1.2 is redundant, since it becomes a particular instance of the second; when the doctrine has weak power objects, for every object A in $\mathbb{C}$ each element ϕ in $P(A)$ determines a term of type $\pi 1$ via the following isomorphism:

$$\begin{tikzcd} 1 \times \pi 1 \arrow[r, "i"] & \pi 1 \\ 1 \times A \arrow[u, "id_1 \times \{j^*\phi\}"] \arrow[r, "j"'] & A \arrow[u, "\{\phi\}"'] \end{tikzcd}$$

we will denote with ϵ_1 the element $(i^{-1})^* \in_1$; in the case $\mathbb{C}$ has all comprehensions, defined to be those morphisms $\lfloor \phi \rfloor\colon X \longrightarrow A$ which are terminal with respect to the property that $\top_X \leq \lfloor \phi \rfloor^*(\phi)$, for every ϕ in A (see [10]), then $\lfloor \phi \rfloor$ is weakly classified by $\{\phi\}$, where the true arrow is $\lfloor \epsilon_1 \rfloor\colon 1 \longrightarrow \pi 1$.

There are several examples of doctrines, we list a few.

Example 1.7. (**Syntactic**) Given a theory $\mathcal{T}$ in a first order language $\mathcal{L}$, the base category $\mathbb{V}$ has lists of distinct variables $\vec{x} = (x_1, x_2, ...x_n)$ as objects and lists of substitutions $[\vec{t}/\vec{y}]\colon \vec{x} \longrightarrow \vec{y}$ as morphisms. Composition is given by simultaneous substitution. For an object $\vec{x}$ in $\mathbb{V}$, the fiber over $\vec{x}$ consists of equivalence classes of well-formed formulae of $\mathcal{L}$ with no more free variables than $x_1, x_2, ...x_n$, with respect to reciprocal entailment of $\mathcal{T}$, see [10].

Example 1.8. (**Subobjects**) Suppose $\mathbb{C}$ a small category with binary products and pullbacks. Consider the functor that assigns for every object A in $\mathbb{C}$ the collection $\mathbf{Sub}(A)$ of subojects with codomain A, ordered by factorization. The top element is (the equivalence class of) the identity arrow. A representative of $\alpha \wedge \beta$ is any pullback of α along β. Given a morphism f in $\mathbb{C}$, $f^*\alpha$ is the class of any pullback of α along f. If $\mathbb{C}$ is regular, the doctrine has left adjoints of all reindexing functors. It is elementary with $\delta_A = \Delta_A\colon A \longrightarrow A \times A$. $\mathbf{Sub}\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$ has full comprehensions. An element $\alpha\colon X \longrightarrow A$ in $\mathbf{Sub}(A)$ has itself as its own comprehension. Consider the following diagrams

$$\begin{tikzcd} X \arrow[r, "\top_X"] \arrow[d, "\top_X"'] & X \arrow[d, "\alpha"] \\ X \arrow[r, "\alpha"'] & A \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "k"] \arrow[dr, "\top_Y"'] & P \arrow[r, "p"] \arrow[d, "f^*\alpha"] & X \arrow[d, "\alpha"] \\ & Y \arrow[r, "f"'] & A \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "h"] \arrow[dr, "\top_X"'] & Q \arrow[r, "q"] \arrow[d, "\alpha^*\beta"] & X \arrow[d, "\beta"] \\ & X \arrow[r, "\alpha"'] & A \end{tikzcd}$$

where P is a pullback of α along f and Q the pullback of β along α. The left one is a pullback and says that $\top_X \le \alpha^*\alpha$. The second proves that if $\top_Y \le f^*\alpha$ ($\le$ is k), then f factorizes through α. Third pullback shows that if $\top_X \simeq \alpha^*\alpha \le \alpha^*\beta$ ($\le$ is h), then $\alpha \le \beta$ ($\le$ is $q \circ h$).
A particular case is when $\mathbb{C}$ is a small, full subcategory of **Set** closed under binary products and subsets and the functor $\mathbf{Sub}\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$ coincides with the powerset functor

Example 1.9. (**Triposes**) We refer to the definition given by Pitts in [11].
Given a category $\mathbb{C}$ with binary products, a tripos is a primary doctrine $P\colon \mathbb{C}^{op} \longrightarrow$ **ISL** such that: (i) for every object A in $\mathbb{C}$, $P(A)$ is a Heyting Algebra (ii) for every arrow f in $\mathbb{C}$, f^* is an homomorphism of Heyting algebras (iii) for every projection arrow π in $\mathbb{C}$ the functor π^* has left and right adjoints satisfying the Beck-Chevalley conditions (iv) P has weak power objects (v) for every object A in $\mathbb{C}$ there exists an element δ_A in $P(A \times A)$ such that, for all α in $P(A\times A)$, $\top_A \le \Delta_A^*(\alpha)$ if and only if $\delta_A \le \alpha$.

All triposes are universal doctrine with weak power objects. They are elementary, since the assignment $\exists_{\Delta_X}(\alpha) := \pi_1^*(\alpha) \wedge \delta_X$ provides a left adjoint to Δ_X^*, in fact

$$
\begin{array}{c}
\alpha \le \Delta_X^*(\beta) \\ \hline
\top_X \le \alpha \Rightarrow \Delta_X^*(\beta) \\ \hline
\top_X \le \Delta_X^*(\pi_1^*(\alpha) \Rightarrow \beta) \\ \hline
\delta_X \le \pi_1^*(\alpha) \Rightarrow \beta \\ \hline
\delta_X \wedge \pi_1^*(\alpha) \le \beta
\end{array}
\qquad
\begin{array}{c}
\exists_{\Delta_X}(\alpha) \le \beta \\ \hline
\pi_1^*(\alpha) \wedge \delta_X \le \beta \\ \hline
\alpha \wedge \Delta_X^*(\delta_X) \le \Delta_X^*(\beta) \\ \hline
\alpha \wedge \top_X \le \Delta_X^*(\beta) \\ \hline
\alpha \le \Delta_X^*(\beta)
\end{array}
$$

and the assignment $\exists_e(\alpha) := \langle \pi_1, \pi_2\rangle^*(\alpha) \wedge \langle \pi_2, \pi_3\rangle^*(\delta_A)$ determines a left adjoint to the reindexing of $e := id_X \times \Delta_A \colon X \times A \longrightarrow X \times A \times A$

$$
\begin{array}{c}
\exists_e(\alpha) \le \beta \\ \hline
e^*\exists_e(\alpha) \le e^*(\beta) \\ \hline
\alpha \wedge \langle \pi_2, \pi_2\rangle^*(\delta_A) \le e^*(\beta) \\ \hline
\alpha \wedge \pi_2^*\Delta_A^*\exists_{\Delta_A}(\top_A) \le e^*(\beta) \\ \hline
\alpha \wedge \pi_2^*(\top_A) \le e^*(\beta) \\ \hline
\alpha \le e^*(\beta)
\end{array}
\qquad
\begin{array}{c}
\alpha \le e^*(\beta) \\ \hline
\top_{X\times A} \le \alpha \Rightarrow e^*(\beta) \\ \hline
\pi_2^*\top_A \le e^*\langle \pi_1, \pi_2\rangle^*(\alpha) \Rightarrow e^*(\beta) \\ \hline
\top_A \le \forall_{\pi_2} e^*(\langle \pi_1, \pi_2\rangle^*(\alpha) \Rightarrow \beta) \\ \hline
\top_A \le \Delta_A^*\forall_{\langle \pi_2, \pi_3\rangle}(\langle \pi_1, \pi_2\rangle^*(\alpha) \Rightarrow \beta) \\ \hline
\delta_A \le \forall_{\langle \pi_2, \pi_3\rangle}(\langle \pi_1, \pi_2\rangle^*(\alpha) \Rightarrow \beta) \\ \hline
\langle \pi_2, \pi_3\rangle^*(\delta_A) \le \langle \pi_1, \pi_2\rangle^*(\alpha) \Rightarrow \beta \\ \hline
\langle \pi_2, \pi_3\rangle^*(\delta_A) \wedge \langle \pi_1, \pi_2\rangle^*(\alpha) \le \beta \\ \hline
\exists_e(\alpha) \le \beta
\end{array}
$$

Similarly it can be proved that Frobenius reciprocity is verified (see also [13], pag 60). Two important examples of triposes are $\mathbb{H}^{(-)}$, for a complete Heyting algebra $\mathbb{H}$, and $\mathbb{P}(\mathcal{N})^{(-)}$, for a partial combinatory algebra over a set $\mathcal{N}$. In each case $\mathbb{C}$ is **Set**, the category of sets and functions. There is no need for a tripos to have comprehensions. But this is the case for localic triposes $\mathbb{H}^{(-)}$ and realizability triposes $\mathbb{P}(\mathcal{N})^{(-)}$. Take a set X and an object $\phi\colon X \longrightarrow \mathbb{H}$: a comprehension of ϕ is given by the inclusion $\lfloor \phi \rfloor\colon \{x \in X \mid \top \le \phi(x)\} \hookrightarrow X$. The same holds for realizability troposes, for which $\lfloor \phi \rfloor\colon \{x \in X \mid \mathcal{N} \subseteq \phi(x)\} \hookrightarrow X$. These comprehensions can not be full. Take $\lfloor \phi \rfloor\colon A \hookrightarrow X$ and consider the

function $\psi \colon X \longrightarrow \mathbb{H}$ defined by $\psi(x) = \top$ if $x \in A$ and $\perp$ otherwise. For this function certainly holds $\lfloor \phi \rfloor^*(\psi) = \top$, but it is not the case that $\phi \leq \psi$.

Example 1.10. (**Topologies**) Consider the category **TOP** of topological spaces and continuos functions. For every topological space X, $\mathcal{O}(X)$ is the collection of its open sets, and then it possesses finite meets and arbitrary joins. Take the functor $\mathcal{O} \colon \mathbf{TOP}^{op} \longrightarrow \mathbf{ISL}$ determined by the following assignment

$$\begin{tikzcd} (X, \mathcal{O}(X)) \arrow[d, "f"'] & & \mathcal{O}(X) \\ (Y, \mathcal{O}(Y)) & \mapsto & \mathcal{O}(Y) \arrow[u, "f^{-1}"'] \end{tikzcd}$$

Even though each fiber is an Heyting algebra, and therefore it has pseudo relative complements (see [12], page 51), $\mathcal{O}$ is not implicational as a doctrine: given a generic continuos function f, we have that pseudo relative complements need not commute with reindexing (see [4], page 39). $\mathcal{O}$ is existential, since every projection functor has a left adjoint (see [12], page 58) satisfying Beck-Chevalley condition and Frobenius reciprocity (recall that projections are open functions). $\mathcal{O}$ has full comprehensions. Given a set X, for any open set S in $O(X)$, define its comprehension to be the inclusion function $\lfloor S \rfloor \colon (S, O_S(X)) \hookrightarrow (X, O(X))$, where $O_S(X)$ is the topology induced by S. These comprehensions are also full. Suppose Q in $O(X)$ such that $\lfloor S \rfloor^{-1}(Q) = S$, this means $\{x \in S \mid x \in Q\} = S \cap Q = S$, so $S \subseteq Q$.

The doctrine has weak power objects. We call Σ the Sierpinski space consisting of two points 0 and 1 and a third non trivial open set $\{1\}$. If a topological space T is locally compact, then there exists in **TOP** the function space Σ^T (see [2] and [5]). Σ extends the subobjects classifier from **Set** to **TOP** in the sense that for every ϕ, open set of T, the characteristic function of the inclusion $\lfloor \phi \rfloor$ is the unique arrow making the following a pullback

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, "\lfloor \phi \rfloor"'] & 1 \arrow[d, "\top"] \\ T \arrow[r, "\chi_\phi"'] & \Sigma \end{tikzcd}$$

for which it holds that $\chi_\phi^{-1}(\{1\}) = \phi$. Now for every topological space A consider any construction that produces a larger locally compact space $\tilde{A}$ such that the inclusion morphism $i_A \colon A \hookrightarrow \tilde{A}$ is continuos and open, e.g. Alexandroff compactifications, see [5]; the following lemma holds: if $f \colon A \times B \longrightarrow \Sigma$ is continuous, then the extension $\tilde{f} \colon \tilde{A} \times B \longrightarrow \Sigma$ is continuous, where $\tilde{f}(a, b) = f(a, b)$ if $a \in A$, then $\tilde{f}(a, b) = 0$.

To prove the lemma it suffices to note that there are no open sets in Σ containing the point 0 other than the top element, then $\tilde{f}^{-1}(\{1\}) = f^{-1}(\{1\})$ and the inclusion function is open. Note that $f = \tilde{f} \circ (i_A \times id)$. Now consider the diagram

$$\begin{tikzcd} A \times \Sigma^{\tilde{A}} \arrow[r, hook, "i_A \times id"] & \tilde{A} \times \Sigma^{\tilde{A}} \arrow[r, "ev_{\tilde{A}}"] & \Sigma \\ A \times B \arrow[u, "id_A \times \overline{\tilde{\chi}_\phi}"] \arrow[r, hook, "i_A \times id"'] & \tilde{A} \times B \arrow[u, "id_{\tilde{A}} \times \overline{\tilde{\chi}_\phi}"'] \arrow[ur, "\tilde{\chi}_\phi"'] & \end{tikzcd}$$

define $\in_A := (ev_{\tilde{A}} \circ (i_A \times id))^{-1}(\{1\})$ and for every open set ϕ in $A \times B$ define $\{\phi\} := \overline{\tilde{\chi}_\phi}$ the exponential transpose of the extension of χ_ϕ. $(id_A \times \overline{\tilde{\chi}_\phi})^{-1}(\in_A) = \chi_\phi^{-1}(\{1\}) = \phi$.

The doctrine fails to be elementary. Given a topological space X, we have that δ_X should be the smallest open set U of $X \times X$ such that $X \subseteq \Delta^{-1}(U)$. In other words

$$\delta_X = (\bigcap_{X \subseteq \Delta_X^{-1}(U)} U)^o$$

if X is the interval $[0,1]$ with the euclidean topology, then δ_X would be empty.

2 Quotients and descents

Recall a construction presented in [9, 10], which is based on the notion of equivalence relation in a doctrine.

Definition 2.1. Given a primary doctrine $P: \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$ and an object A of $\mathbb{C}$, an element ρ in $P(A \times A)$ is said to be an **equivalence relation** on A if

reflexivity: $\top_A \leq \Delta_A^*(\rho)$

symmetry: $\rho \leq \langle \pi_1, \pi_2 \rangle^*(\rho)$

transitivity: $\langle \pi_1, \pi_2 \rangle^*(\rho) \wedge \langle \pi_2, \pi_3 \rangle^*(\rho) \leq \langle \pi_1, \pi_3 \rangle^*(\rho)$

Note that if the doctrine P is also elementary, then δ_A is an equivalence relation on A for every object A in $\mathbb{C}$.

In [9, 10] the authors consider a certain category $\mathcal{Q}_P$, when $P: \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$ is elementary. In the category $\mathcal{Q}_P$

objects are pairs (A, ρ) such that ρ is an equivalence relation on A

morphisms $f: (A, \rho) \longrightarrow (B, \sigma)$ are arrows $f: A \longrightarrow B$ in $\mathbb{C}$ such that $\rho \leq (f \times f)^*\sigma$

and composition is given as in $\mathbb{C}$.

A first remark is that the construction gives a category in the more general case of P primary. The category $\mathcal{Q}_P$ has binary products: given (A, ρ) and (B, σ) in $\mathcal{Q}_P$, $(A, \rho) \times (B, \sigma) := (A \times B, \rho \boxtimes \sigma)$, where $\rho \boxtimes \sigma$ is $\langle \pi_1, \pi_3 \rangle^* \rho \wedge \ \langle \pi_2, \pi_4 \rangle^* \sigma$. Moreover if $\mathbb{C}$ has a terminal object, $\mathcal{Q}_P$ has a terminal object.

There is an obvious forgetful functor $\mathbb{U}: \mathcal{Q}_P \longrightarrow \mathbb{C}$, and a functor $\nabla: \mathbb{C} \longrightarrow \mathcal{Q}_P$, determined by the following assignments

$$(\mathbb{U}) \quad \begin{array}{c}(A,\rho)\\ \downarrow f\\ (B,\sigma)\end{array} \mapsto \begin{array}{c}A\\ \downarrow f\\ B\end{array} \qquad\qquad (\nabla) \quad \begin{array}{c}A\\ \downarrow f\\ B\end{array} \mapsto \begin{array}{c}(A,\delta_A)\\ \downarrow f\\ (B,\delta_B)\end{array}$$

∇ is clearly a functor since, for every morphism f in $\mathbb{C}$, $\delta_A \leq (f \times f)^*\delta_B$.

Lemma 2.2. Given an elementary doctrine $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, the functor ∇ is left adjoint to $\mathbb{U}$.

Proof. For every object (B, σ) in $\mathcal{Q}_P$, the map $\varepsilon_B := id_B\colon (B, \delta_B) \longrightarrow (B, \sigma)$ is the B-component of a natural transformation. This is the counite of the adjunction, since for every object A in $\mathbb{C}$ and every arrow $f\colon (A, \delta_A) \longrightarrow (B, \sigma)$ in $\mathcal{Q}_P$ the diagram commutes

$$\begin{tikzcd}
(B,\delta_B) \arrow[r, "id_B"] & (B,\sigma) \\
(A,\delta_A) \arrow[u, "f"] \arrow[ur, "f"'] &
\end{tikzcd}$$

and f is the unique such arrow. $\square$

Definition 2.3. Given a primary doctrine $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$ and an equivalence relation ρ on an object A of $\mathbb{C}$, the poset of descent data $\mathcal{D}es_\rho$ is the sub-order of $P(A)$ made by those α such that

$$\pi_1^*(\alpha) \ \wedge \ \rho \ \leq \ \pi_2^*(\alpha)$$

The order $\mathcal{D}es_\rho$ is closed under meets and it has trivially $\top_A$, then $\mathcal{D}es_\rho$ is an inf-semilattice.
The following proposition generalizes to primary doctrines a similar result given for elementary doctrine in [9, 10].

Proposition 2.4. Given a primary doctrine $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, the assignment

$$\begin{tikzcd}
(A,\rho) \arrow[d, "f"'] & & \mathcal{D}es_\rho \\
(B,\sigma) & \mapsto & \mathcal{D}es_\sigma \arrow[u, "f^*"']
\end{tikzcd}$$

determines a primary doctrine $P_{\mathcal{D}}\colon \mathcal{Q}_P^{op} \longrightarrow \mathbf{ISL}$.

Proof. It suffices to note that, for every β in $\mathcal{D}es_\sigma$, $f^*\beta$ is in $\mathcal{D}es_\rho$, that can be proved by taking the descent condition on β, applying to both sides $(f \times f)^*$ and use the fact that $\rho \ \leq \ (f \times f)^*\sigma$. $\square$

3 A co-free construction

There is an obvious forgetful functor $\mathcal{U}\colon \mathbf{ED} \longrightarrow \mathbf{PD}$, which maps every elementary doctrine to itself. We shall show that the construction in 2 extends to a 2-right adjoint to it.

The following lemma is a strengthening of a similar result in [9].

Lemma 3.1. Given a primary doctrine $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, the doctrine $P_{\mathcal{D}}\colon \mathcal{Q}_P^{op} \longrightarrow \mathbf{ISL}$ built as in 2.4 is elementary.

Proof. Consider (A, ρ) in $\mathcal{Q}_P$. Note that ρ is an element of $\mathcal{D}es_{\rho\boxtimes\rho}$, since

$$\pi_1^*\rho \wedge (\rho \boxtimes \rho) = \langle\pi_1, \pi_2\rangle^*\rho \wedge\ \langle\pi_1, \pi_3\rangle^*\rho \wedge\ \langle\pi_2, \pi_4\rangle^*\rho$$

and by transitivity of ρ

$$\pi_1^*\rho \wedge \rho \boxtimes \rho\ \leq \langle\pi_3, \pi_4\rangle^*\rho\ =\ \pi_2^*\rho$$

Let $\delta_{(A,\rho)}$ be ρ and define $\exists_{\Delta_A}\alpha := \pi_1^*\alpha\ \wedge \rho$. We want to prove that, for every α in $\mathcal{D}es_\rho$ and β in $\mathcal{D}es_{\rho\boxtimes\rho}$, $\exists_{\Delta_A}\alpha \leq \beta$ if and only if $\alpha \leq \Delta_A^*\beta$. Suppose $\exists_{\Delta_A}\alpha \leq \beta$, which means $\pi_1^*(\alpha)\ \wedge \rho\ \leq \beta$, and apply Δ_A^* to both sides, to obtain $\alpha \wedge \Delta_A^*\rho\ \leq \Delta_A^*\beta$. So $\alpha \leq \Delta_A^*\beta$, by reflexivity of ρ. Assume now $\alpha \leq \Delta_A^*\beta$, the descent condition for β gives:

$$\langle\pi_1, \pi_2\rangle^*\beta \wedge \langle\pi_1, \pi_3\rangle^*\rho \wedge\ \langle\pi_2, \pi_4\rangle^*\rho\ \leq\ \langle\pi_3, \pi_4\rangle^*\beta$$

By reindexing along $(\Delta_A \times id_A \times id_A)^*$ and $(\Delta_A \times id_A)^*$ one obtains

$$\pi_1^*\Delta_A^*\beta\ \wedge \rho\ \leq \beta$$

by reflexivity of ρ

$$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \leq \Delta_A^*\beta}{\pi_1^*\alpha \leq \pi_1^*\Delta_A^*\beta}}{\pi_1^*\alpha\ \wedge\ \rho\ \leq\ \pi_1^*\Delta_A^*\beta\ \wedge\ \rho} \qquad \pi_1^*\Delta_A^*\beta\ \wedge \rho\ \leq \beta}{\pi_1^*\alpha\ \wedge\ \rho\ \leq\ \beta}}{\exists_{\Delta_A}\alpha\ \leq\ \beta}$$

To verify the conditions ii) of 1.2, consider an object (X, τ) and let $e := id_X \times \Delta_A$ be a morphism in $\mathcal{Q}_P$. The proof that if $\exists_e(\alpha) \leq \beta$, then $\alpha \leq e^*(\beta)$ is similar to that in example 1.9 (where ρ is δ_A). The proof of the converse, is essentially as before where:

$$\langle\pi_1, \pi_2, \pi_3, \rangle^*\beta\ \wedge\ \langle\pi_1, \pi_4\rangle^*\tau\ \wedge\ \langle\pi_2, \pi_5\rangle^*\rho\ \wedge\ \langle\pi_3, \pi_6\rangle^*\rho\ \leq\ \langle\pi_4, \pi_5, \pi_6\rangle^*\beta$$

and reindexing along the following composition

$$\begin{tikzcd}
X \times A \times A \arrow[d, "id_X \times \Delta_A \times id_A"] & & X \times A \times A \times X \times A \times A \\
X \times A \times A \times A \arrow[dr, bend right, "\Delta_X \times id_A \times id_A \times id_A"] & & X \times A \times X \times A \times A \arrow[u, "id_X \times \Delta_A \times id_X \times id_A \times id_A"'] \\
& X \times X \times A \times A \times A \arrow[ur, bend right, "id_X \times tw \times id_A \times id_A"] &
\end{tikzcd}$$

$\square$

Given a 1-morphism in **PD**, $(F, f)\colon P \longrightarrow R$, consider the functor $F_\mathcal{D}$ defined by the following assignment

$$\begin{tikzcd}
(A, \rho) \arrow[d, "q"'] & & (FA, \langle\pi_1, \pi_2\rangle^* f_{A\times A}(\rho)) \arrow[d, "Fq"] \\
(B, \sigma) \arrow[r, phantom, "\mapsto"] & & (FB, \langle\pi_1, \pi_2\rangle^* f_{B\times B}(\sigma))
\end{tikzcd}$$

and the $\mathcal{Q}_P$-indexed family of arrow $f_\mathcal{D}$ whose (A, ρ)-component is the restriction of $f_A\colon P(A) \longrightarrow R(FA)$ to $\mathcal{D}es_\rho$

Lemma 3.2. Given a 1-morphism in **PD**, $(F,f)\colon P \longrightarrow R$ the pair $(F_{\mathcal{D}}, f_{\mathcal{D}})\colon P_{\mathcal{D}} \longrightarrow R_{\mathcal{D}}$ determines a 1-morphism in **ED**.

Proof. First note that $\langle \pi_1, \pi_2\rangle^* f_{A\times A}(\rho)$ is an equivalence relation since ρ is and f is natural. Fq is a morphism in $\mathcal{Q}_P$, since $\langle \pi_1, \pi_2\rangle^* f_{A\times A}(\rho) \leq (Fq \times Fq)^*\langle \pi_1, \pi_2\rangle^* f_{B\times B}(\sigma) = \langle \pi_1, \pi_2\rangle^* F(q\times q)^* f_{B\times B}(\sigma) = \langle \pi_1, \pi_2\rangle^* f_{A\times A}(q\times q^*\sigma)$, for naturality of f. It is left to show that the images of the restriction is $\mathcal{D}es_{\langle \pi_1,\pi_2\rangle^* f_{A\times A}(\rho)}$, but this is true since, for α in $\mathcal{D}es_\rho$, $\pi_1^*\alpha \ \wedge \rho \ \leq \pi_2^*\alpha$, then apply $f_{A\times A}$ to both sides and, recalling that $f_{A\times A}\circ \pi_1^* = \pi_1^* \circ f_A$ for naturality of f, one has $\pi_1^* f_A^*\alpha \ \wedge \ f_{A\times A}(\rho) \ \leq \ \pi_2^* f_A^*\alpha$. Now it suffices to reindex both sides along $\langle \pi_1, \pi_2\rangle$. The last step is to show that $f_{\mathcal{D}}$ preserves the elementary structure, i.e. $f_{\mathcal{D}(A,\rho)\times(A,\rho)}(\delta_{(A,\rho)}) = \langle F_{\mathcal{D}}\pi_1, F_{\mathcal{D}}\pi_2\rangle^*(\delta_{F_{\mathcal{D}}(A,\rho)})$, which reduces to the following equality $f_{A\times A}(\rho) = \langle F\pi_1, F\pi_2\rangle^*(\langle \pi_1,\pi_2\rangle^* f_{A\times A}(\rho))$, where $\langle \pi_1,\pi_2\rangle \circ \langle F\pi_1, F\pi_2\rangle = id_{F(A\times A)}$. ☐

Consider the functor $(-)_{\mathcal{D}}\colon \mathbf{PD} \longrightarrow \mathbf{ED}$

$$\begin{tikzcd} P \arrow[d, "{(F,f)}"'] & \mapsto & P_{\mathcal{D}} \arrow[d, "{(F_{\mathcal{D}}, f_{\mathcal{D}})}"] \\ R & & R_{\mathcal{D}} \end{tikzcd}$$

For every doctrine $P\colon \mathbb{C} \longrightarrow \mathbf{ISL}$ in **PD** there is a 1-morphism ε_P from $P_{\mathcal{D}}$ to P given by the pair $(\mathbb{U}, i)$, where $\mathbb{U}\colon \mathcal{Q}_P \longrightarrow \mathbb{C}$ is the forgetful functor defined before 2.2, while the A-component of i is the inclusion functor $\mathcal{D}es_\rho \hookrightarrow P(A)$.

Proposition 3.3. The natural transformation ε is the counit of an adjunction $\mathcal{U} \dashv (-)_{\mathcal{D}}$.

Proof. Note that $\mathcal{U}(P) = P$; given an elementary doctrine $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, a morphism $(F,f)\colon \mathcal{U}(P) \longrightarrow R$ in **PD**, consider the arrow $(\overline{F}, \overline{f})\colon P \longrightarrow R_{\mathcal{D}}$ in **ED**, determined by the following composition

$$\begin{tikzcd} \mathbb{C}^{op} \arrow[d, "\nabla"'] \arrow[rr, bend left, "P"{name=A}] & & \\ \mathcal{Q}_P^{op} \arrow[rr, "P_{\mathcal{D}}"'{name=B}] \arrow[d, "F_{\mathcal{D}}"'] & & \mathbf{ISL} \\ \mathcal{Q}_R^{op} \arrow[rru, bend right, "R_{\mathcal{D}}"'{name=C}] & & \arrow[from=A, to=B, Rightarrow, "id_{PA}"] \arrow[from=B, to=C, Rightarrow, "f_{\mathcal{D}}"] \end{tikzcd}$$

then $\overline{F} := F_{\mathcal{D}} \circ \nabla$ and $\overline{f} := f_{\mathcal{D}} \circ id_{PA}$. Where the natural transformation $P \longrightarrow P_{\mathcal{D}} \circ \nabla$ is the identity from the fact that $\mathcal{D}es_{\delta_A} = P(A)$. What is left to prove is that $(\overline{F}, \overline{f})$ is the unique arrow that makes the following diagram commutes

$$\begin{tikzcd} \mathcal{Q}_R^{op} \arrow[rr, bend left, "{(\mathbb{U},i)}"] \arrow[dr, "R_{\mathcal{D}}"] & & \mathbb{D}^{op} \arrow[dl, "R"'] \\ & \mathbf{ISL} & \\ & \mathbb{C}^{op} \arrow[u, "P"] \arrow[uul, bend left, "{(\overline{F},\overline{f})}"] \arrow[uur, bend right, "{(F,f)}"'] & \end{tikzcd}$$

Commutativity: recall that, for an object A in $\mathbb{C}$, $\mathbb{U}(\overline{F})(A)$ is $\mathbb{U}(F_{\mathcal{D}}(\nabla(A)))$, then follow the assignments below

$$A \mapsto (A, \delta_A) \mapsto (FA, \delta_{FA}) \mapsto FA$$

moreover $(i \circ \overline{f})_A$ is $i_A \circ f_{\mathcal{D}A} \circ id_{PA}$, then take α in $P(A)$ and follow the assignments

$$\alpha \mapsto f_A(\alpha) \mapsto i(f_A(\alpha)) = f_A(\alpha)$$

Uniqueness is given by the fact that $(\mathbb{U}, i)$ is mono, since $\mathbb{U}$ is the identity on objects and morphism and i is an inclusion functor. $\square$

4 A co-free construction

There is an obvious forgetful functor $\mathcal{U}\colon \mathbf{ED} \longrightarrow \mathbf{PD}$, which maps every elementary doctrine to itself. We shall show that the construction in 2 extends to a 2-right adjoint to it.

The following lemma is a strengthening of a similar result in [9].

Lemma 4.1. Given a primary doctrine $P\colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, the doctrine $P_{\mathcal{D}}\colon \mathcal{Q}_P^{op} \longrightarrow \mathbf{ISL}$ built as in 2.4 is elementary.

Proof. Consider (A, ρ) in $\mathcal{Q}_P$. Note that ρ is an element of $\mathcal{D}es_{\rho\boxtimes\rho}$, since

$$\pi_1^*\rho \wedge (\rho \boxtimes \rho) = \langle \pi_1, \pi_2\rangle^*\rho \wedge \ \langle \pi_1, \pi_3\rangle^*\rho \wedge \ \langle \pi_2, \pi_4\rangle^*\rho$$

and by transitivity of ρ

$$\pi_1^*\rho \wedge \rho \boxtimes \rho \ \leq \langle \pi_3, \pi_4\rangle^*\rho \ = \ \pi_2^*\rho$$

Let $\delta_{(A,\rho)}$ be ρ and define $\exists_{\Delta_A}\alpha := \pi_1^*\alpha \ \wedge \rho$. We want to prove that, for every α in $\mathcal{D}es_\rho$ and β in $\mathcal{D}es_{\rho\boxtimes\rho}$, $\exists_{\Delta_A}\alpha \leq \beta$ if and only if $\alpha \leq \Delta_A^*\beta$. Suppose $\exists_{\Delta_A}\alpha \leq \beta$, which means $\pi_1^*(\alpha) \ \wedge \rho \ \leq \beta$, and apply Δ_A^* to both sides, to obtain $\alpha \wedge \Delta_A^*\rho \ \leq \Delta_A^*\beta$. So $\alpha \leq \Delta_A^*\beta$, by reflexivity of ρ. Assume now $\alpha \leq \Delta_A^*\beta$, the descent condition for β gives:

$$\langle \pi_1, \pi_2\rangle^*\beta \wedge \langle \pi_1, \pi_3\rangle^*\rho \wedge \ \langle \pi_2, \pi_4\rangle^*\rho \ \leq \ \langle \pi_3, \pi_4\rangle^*\beta$$

By reindexing along $(\Delta_A \times id_A \times id_A)^*$ and $(\Delta_A \times id_A)^*$ one obtains

$$\pi_1^*\Delta_A^*\beta \ \wedge \rho \ \leq \beta$$

by reflexivity of ρ

$$\dfrac{\dfrac{\dfrac{\alpha \leq \Delta_A^*\beta}{\pi_1^*\alpha \leq \pi_1^*\Delta_A^*\beta}}{\pi_1^*\alpha \ \wedge \ \rho \ \leq \ \pi_1^*\Delta_A^*\beta \ \wedge \ \rho} \qquad \pi_1^*\Delta_A^*\beta \ \wedge \rho \ \leq \beta}{\dfrac{\pi_1^*\alpha \ \wedge \ \rho \ \leq \ \beta}{\exists_{\Delta_A}\alpha \ \leq \ \beta}}$$

To verify the conditions ii) of 1.2, consider an object (X, τ) and let $e := id_X \times \Delta_A$ be a morphism in $\mathcal{Q}_P$. The proof that if $\exists_e(\alpha) \leq \beta$, then $\alpha \leq e^*(\beta)$ is similar

to that in example 1.9 (where ρ is δ_A). The proof of the converse, is essentially as before where:

$$\langle \pi_1, \pi_2, \pi_3, \rangle^* \beta \ \wedge \ \langle \pi_1, \pi_4 \rangle^* \tau \ \wedge \ \langle \pi_2, \pi_5 \rangle^* \rho \ \wedge \ \langle \pi_3, \pi_6 \rangle^* \rho \ \leq \ \langle \pi_4, \pi_5, \pi_6 \rangle^* \beta$$

and reindexing along the following composition

$$\begin{tikzcd}
X \times A \times A \arrow[d, "id_X \times \Delta_A \times id_A"] & & X \times A \times A \times X \times A \times A \\
X \times A \times A \times A \arrow[dr, bend right, "\Delta_X \times id_A \times id_A \times id_A"] & & X \times A \times X \times A \times A \arrow[u, "id_X \times \Delta_A \times id_X \times id_A \times id_A"'] \\
& X \times X \times A \times A \times A \arrow[ur, bend right, "id_X \times tw \times id_A \times id_A"] &
\end{tikzcd}$$

□

Given a 1-morphism in **PD**, $(F, f) \colon P \longrightarrow R$, consider the functor $F_{\mathcal{D}}$ defined by the following assignment

$$\begin{tikzcd}
(A, \rho) \arrow[d, "q"'] & \mapsto & (FA, \langle \pi_1, \pi_2 \rangle^* f_{A \times A}(\rho)) \arrow[d, "Fq"] \\
(B, \sigma) & & (FB, \langle \pi_1, \pi_2 \rangle^* f_{B \times B}(\sigma))
\end{tikzcd}$$

and the $\mathcal{Q}_P$-indexed family of arrow $f_{\mathcal{D}}$ whose (A, ρ)-component is the restriction of $f_A \colon P(A) \longrightarrow R(FA)$ to $\mathcal{D}es_\rho$

Lemma 4.2. Given a 1-morphism in **PD**, $(F, f) \colon P \longrightarrow R$ the pair $(F_{\mathcal{D}}, f_{\mathcal{D}}) \colon P_{\mathcal{D}} \longrightarrow R_{\mathcal{D}}$ determines a 1-morphism in **ED**.

Proof. First note that $\langle \pi_1, \pi_2 \rangle^* f_{A \times A}(\rho)$ is an equivalence relation since ρ is and f is natural. Fq is a morphism in $\mathcal{Q}_P$, since $\langle \pi_1, \pi_2 \rangle^* f_{A \times A}(\rho) \leq (Fq \times Fq)^* \langle \pi_1, \pi_2 \rangle^* f_{B \times B}(\sigma) = \langle \pi_1, \pi_2 \rangle^* F(q \times q)^* f_{B \times B}(\sigma) = \langle \pi_1, \pi_2 \rangle^* f_{A \times A}(q \times q^* \sigma)$, for naturality of f. It is left to show that the images of the restriction is $\mathcal{D}es_{\langle \pi_1, \pi_2 \rangle^* f_{A \times A}(\rho)}$, but this is true since, for α in $\mathcal{D}es_\rho$, $\pi_1^* \alpha \ \wedge \rho \ \leq \pi_2^* \alpha$, then apply $f_{A \times A}$ to both sides and, recalling that $f_{A \times A} \circ \pi_1^* = \pi_1^* \circ f_A$ for naturality of f, one has $\pi_1^* f_A^* \alpha \ \wedge \ f_{A \times A}(\rho) \ \leq \ \pi_2^* f_A^* \alpha$. Now it suffices to reindex both sides along $\langle \pi_1, \pi_2 \rangle$. The last step is to show that $f_{\mathcal{D}}$ preserves the elementary structure, i.e. $f_{\mathcal{D}(A,\rho) \times (A,\rho)}(\delta_{(A,\rho)}) = \langle F_{\mathcal{D}} \pi_1, F_{\mathcal{D}} \pi_2 \rangle^* (\delta_{F_{\mathcal{D}}(A,\rho)})$, which reduces to the following equality $f_{A \times A}(\rho) = \langle F\pi_1, F\pi_2 \rangle^* (\langle \pi_1, \pi_2 \rangle^* f_{A \times A}(\rho))$, where $\langle \pi_1, \pi_2 \rangle \circ \langle F\pi_1, F\pi_2 \rangle = id_{F(A \times A)}$. □

Consider the functor $(-)_{\mathcal{D}} \colon \mathbf{PD} \longrightarrow \mathbf{ED}$

$$\begin{tikzcd}
P \arrow[d, "(F,f)"'] & \mapsto & P_{\mathcal{D}} \arrow[d, "(F_{\mathcal{D}}, f_{\mathcal{D}})"] \\
R & & R_{\mathcal{D}}
\end{tikzcd}$$

For every doctrine $P \colon \mathbb{C} \longrightarrow \mathbf{ISL}$ in **PD** there is a 1-morphism ε_P from $P_{\mathcal{D}}$ to P given by the pair $(\mathbb{U}, i)$, where $\mathbb{U} \colon \mathcal{Q}_P \longrightarrow \mathbb{C}$ is the forgetful functor defined before 2.2, while the A-component of i is the inclusion functor $\mathcal{D}es_\rho \hookrightarrow P(A)$.

Proposition 4.3. The natural transformation ε is the counit of an adjunction $\mathcal{U} \dashv (-)_{\mathcal{D}}$.

Proof. Note that $\mathcal{U}(P) = P$; given an elementary doctrine $P \colon \mathbb{C}^{op} \longrightarrow \mathbf{ISL}$, a morphism $(F, f) \colon \mathcal{U}(P) \longrightarrow R$ in **PD**, consider the arrow $(\overline{F}, \overline{f}) \colon P \longrightarrow R_{\mathcal{D}}$ in **ED**, determined by the following composition

$$\begin{tikzcd}
\mathbb{C}^{op} \arrow[d, "\nabla"'] \arrow[rrd, bend left, "P"] & {} \arrow[d, phantom, "\Downarrow id_{PA}"] & \\
\mathcal{Q}_P^{op} \arrow[rr, "P_{\mathcal{D}}"'] \arrow[d, "F_{\mathcal{D}}"'] & {} \arrow[d, phantom, "\Downarrow f_{\mathcal{D}}"] & \mathbf{ISL} \\
\mathcal{Q}_R^{op} \arrow[rru, bend right, "R_{\mathcal{D}}"'] & {} &
\end{tikzcd}$$

then $\overline{F} := F_{\mathcal{D}} \circ \nabla$ and $\overline{f} := f_{\mathcal{D}} \circ id_{PA}$. Where the natural transformation $P \longrightarrow P_{\mathcal{D}} \circ \nabla$ is the identity from the fact that $\mathcal{D}es_{\delta_A} = P(A)$. What is left to prove is that $(\overline{F}, \overline{f})$ is the unique arrow that makes the following diagram commutes

$$\begin{tikzcd}
\mathcal{Q}_R^{op} \arrow[rr, bend left, "(\mathbb{U}, i)"] \arrow[rd, "R_{\mathcal{D}}"] & & \mathbb{D}^{op} \arrow[ld, "R"'] \\
& \mathbf{ISL} & \\
& \mathbb{C}^{op} \arrow[u, "P"] \arrow[luu, bend left, "(\overline{F}, \overline{f})"] \arrow[ruu, bend right, "(F, f)"'] &
\end{tikzcd}$$

Commutativity: recall that, for an object A in $\mathbb{C}$, $\mathbb{U}(\overline{F})(A)$ is $\mathbb{U}(F_{\mathcal{D}}(\nabla(A)))$, then follow the assignments below

$$A \mapsto (A, \delta_A) \mapsto (FA, \delta_{FA}) \mapsto FA$$

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$$\alpha \mapsto f_A(\alpha) \mapsto i(f_A(\alpha)) = f_A(\alpha)$$

Uniqueness is given by the fact that $(\mathbb{U}, i)$ is mono, since $\mathbb{U}$ is the identity on objects and morphism and i is an inclusion functor. $\square$

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